Transition to College Mathematics and Statistics

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Unit 2: *Functions Modeling Change*

Transition to College Mathematics and Statistics (TCMS) consists of eight coherent and focused units with deliberate connections among topics across units. Each unit is comprised of two to four problem-based, inquiry-oriented, and technology-rich multi-day lessons. Each lesson consists of two to four related investigations emphasizing mathematical modeling and important mathematical practices and habits of mind.

Units culminate with a "Looking Back" lesson intended for students to review and synthesize their understanding of key ideas developed in the unit. As such, the following "Looking Back" lessons for each TCMS unit provide potential users an overview of our approach to important mathematical ideas and the expectations and nature of collaborative student work. Preceding each "Looking Back" lesson is the table of contents for the unit.

Mc Graw Hill Education

Table of Contents

UNIT 2 Functions Modeling Change

LESSON 4 Looking Back

In this unit, you revisited a variety of basic types of functions and developed strategies for modifying those functions to build models for more complex relationships. As a result of that work, you developed greater skill in using linear, exponential, quadratic, power, and circular functions to model data patterns and problem conditions. You also learned how to modify the rules of basic function types to model and analyze data patterns that are related to familiar functions by vertical and horizontal translation, by reflection across the x-axis, and by vertical and horizontal stretching or compressing.

The tasks in this final lesson give you a chance to review your skill and understanding of function families and transformations.

Thinking in Millennia New Year's Day is celebrated in cultures and countries around the world. But January 1, 2000 was a very special date, because it marked the beginning of a new millennium, or thousand-year time period. The occasion prompted many comparisons with life at the start of the previous millennium in the year 1000. At that time:

- Earth's human population was about 250 million and growing at a rate of 0.1% per year.
- one fourth of the population lived in China, and the world's largest city was Cordoba, Spain, with a population of 450,000.
- half of all children died before the age of five.

By the year 2000, the world's population had increased to about 6 billion (6,000 million) and it was growing at an annual rate of 1.7%.

1. About how many people were added to the world population in the year 1000? In the year 2000?

2 The relatively low world population growth rate in the year 1000 continued until the 1700s, when more modern medicine and improved water and sewage systems emerged.

a. Suppose that world population growth had continued at an annual rate of 0.1% from 1000 to 1700. What function would this condition imply as a model for estimating world population $P(t)$ in year $1000 + t$?

b. What world population does your model in Part a predict for 1700? The actual world population in 1700 is estimated to have been about 640 million.

c. Suppose that world population had continued to increase by the same number of people in each year after 1000. What function would this condition imply as a model for estimating world population in year $1000 + t$?

d. What world population does your model in Part c predict for 1700?

e. If world population growth had continued at the rates in the year 1000 until the year 2000, what would the population models have predicted for the year 2000:

i. using the 0.1% growth rate condition in Part a?

ii. using the constant number-of-people-per-year growth rate condition in Part c?

f. If world population grows beyond the year-2000 figure of 6 billion at the rate of 1.7% per year:

i. what function would this condition imply as a model for predicting world population in year $2000 + t$?

ii. what world population does the model in part i predict for the year 2050?

iii. what reasons can you imagine for doubting that the prediction in part ii will actually occur?

Planetary Motion Whenever scientists report an unusual astronomical event, we are reminded that our Earth is a very small planet in a very large universe. For example, when the Hale-Bopp comet flew within sight of Earth during 1996 and 1997, there was considerable discussion about the chances that other comets and asteroids might actually enter Earth's atmosphere. Some scientists even made estimates of the damage that would result from such an event.

One theory predicts that if an asteroid with a diameter of only 3 miles were to land in the middle of the North Atlantic Ocean, it would send a 300-foot tsunami crashing on the shores of North America and Europe. Fortunately, such an event is estimated to occur only once every 10,000,000 years!

3 Comets and asteroids have irregular shapes, but most can be approximated as spheres.

a. If an asteroid has average diameter d miles, what function rules give the:

i. disk area of the cross section at a diameter of the asteroid?

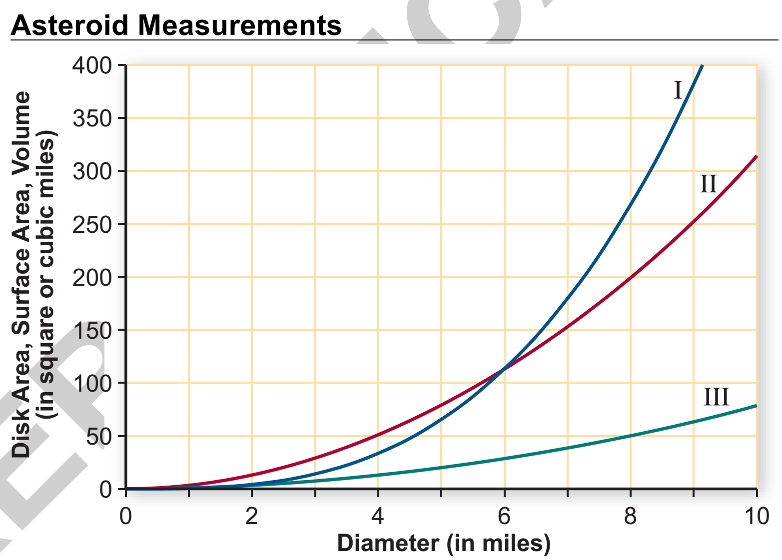
ii. total surface area of the approximately spherical body?

iii. volume of the approximately spherical body?

b. The diagram below shows graphs of the three measurement functions in Part a on the interval $0 \leq d \leq 10$.

i. Match the functions and graphs and explain how you know you are correct.

ii. Explain what the relative shape of the three graphs says about the rates at which disk area, surface area, and volume change as asteroid diameter increases.



c. Comet Hale-Bopp appeared recently near Earth. Prior to that appearance, it last came near Earth over 4,200 years ago. The gravitational pull of the planet Jupiter will cause it to return near Earth again in about 2,400 years. The gravitational attraction of two large bodies is directly proportional to the product of their masses and inversely proportional to the square of the distance between their centers.

i. If m_J and m_H represent the masses of Jupiter and comet Hale-Bopp respectively and d represents the distance between the centers of those masses, what is the form of the rule for the function $g(d)$ telling the gravitational attraction of those bodies at any distance d?

ii. What will a graph of $g(d)$ look like?

4 The visible Moon varies in size from a full moon to a new moon (not visible at all) and back to a full moon in a cycle that takes roughly 30 days. Dates of many important religious and cultural events are set by reference to lunar calendars.

a. What function family seems likely to be the best starting point in building a model that tells visible area at any time during its 30-day cycle of phases:

i. if you assume that the cycle starts with a full moon?

ii. if you assume that the cycle starts with a half-moon on its way toward a full moon?

b. What particular members of the function families described in your response to Part a are likely to be good models of change in the visible moon if we assume that a full moon is 100%, a new moon is 0%, the cycle is 30 days long, and

i. the cycle starts with a full moon?

ii. the cycle starts with a half-moon on its way toward a full moon?

Matching Function Rules and Graphs In the lessons of this unit, you discovered that when building models of data patterns, it helps if you can identify a likely function rule by inspecting the graph.

5 Match each of the functions given in Parts a–j with their graphs in the following diagrams without using technology.

a. $y = x - 2$

b. $y = (x + 2)^2 - 5$

c. $y = 1.5^x - 2$

d. $y = 1.5x - 2$

e. $y = 3 \cos x + 1$

f. $y = 3 \cos x - 1$

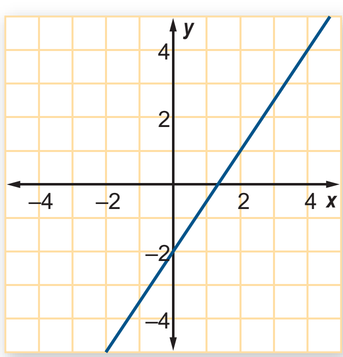
g. $y = (x - 2)^2 - 5$

h. $y = 3 \cos 2x - 1$

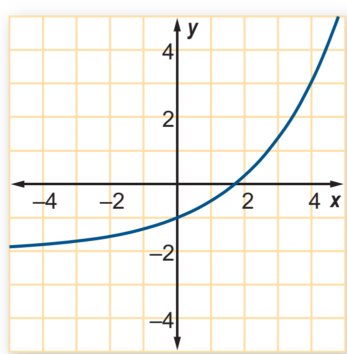
i. $y = -2(x + 1)$

j. $y = 2|x - 1|$

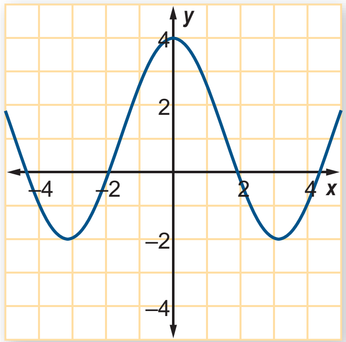
Graph I



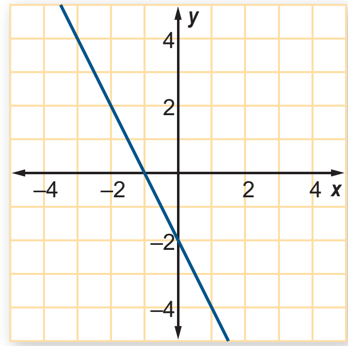
Graph II



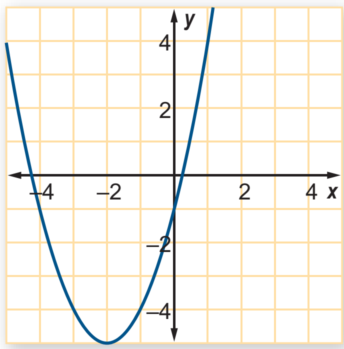
Graph III



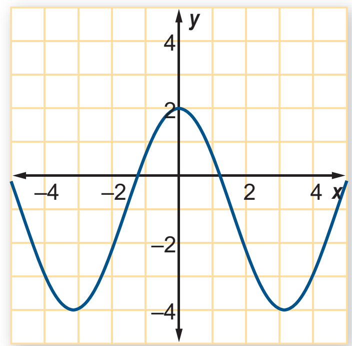
Graph IV



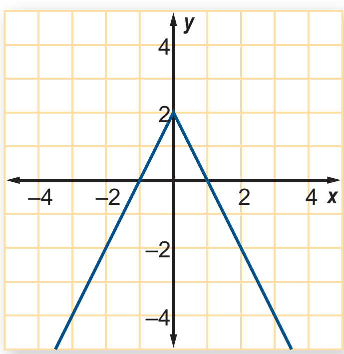
Graph V



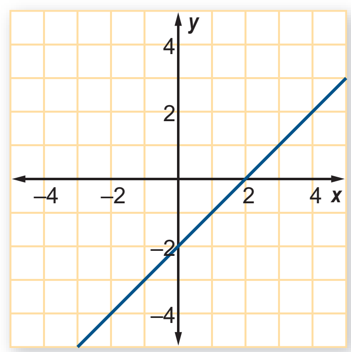
Graph VI



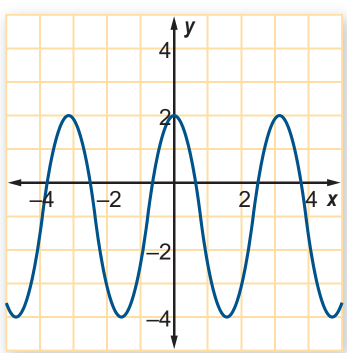
Graph VII



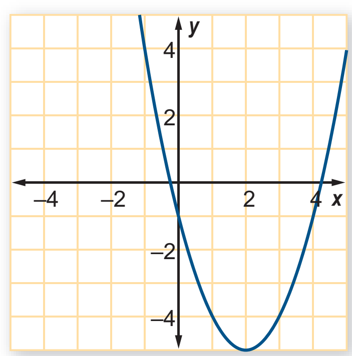
Graph VIII



Graph IX



Graph X



Summarize the Mathematics

In this unit, you investigated a variety of situations in which rules for familiar functions had to be modified to model patterns in data plots and conditions in particular problems.

a. What table and graph patterns and problem conditions are clues to use the following families of functions as models?

   i. Linear
   ii. Exponential
   iii. Quadratic
   iv. Direct power
   v. Inverse power
   vi. Absolute value
   vii. Sine
   viii. Cosine
   ix. Square root
   x. Base-10 logarithm

b. What are the general forms of rules for each of the types of functions listed in Part a. What do the values of the parameters in those rules tell you about the expected patterns in tables and graphs of those particular functions?

c. How can you adjust the rule for a function $f(x)$ so that its graph matches the graphs of a function $g(x)$ related by these transformations?

   i. Vertical translation
   ii. Horizontal translation
   iii. Vertical stretching/compressing
   iv. Horizontal stretching/compressing
   v. Reflection across the x-axis

d. How can the basic sine and cosine functions be modified to give functions with amplitude A and period p?

Be prepared to share your responses and reasoning with the class.

Check Your Understanding

Write, in outline form, a summary of the important mathematical concepts and methods developed in this unit. Organize your summary so that it can be used as a quick reference in future units.